Finishing Bases:

Definition: A *minimal spanning set* of V is a set that spans V but does not span V if a single vector is removed.

So the set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a minimal spanning set of V if $\mathcal{S}$ spans V but the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ do not.

Definition: A *maximal linearly independent* set in V is a set that is LI, but becomes LD if a single additional vector in V is included.

So the set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a maximal linearly independent set in V if $\mathcal{S}$ is LI but the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{x}\}$ is LD for ANY $\mathbf{x} \in V$.

Definition: A *basis* for vector space V is a spanning set for V that is also a linearly independent set.

Definition: If a set of n vectors forms a basis of the vector space V then the *dimension* of V, $\dim(V) = n$.

Turns out these are all the same thing.

Theorem:
The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \in V$ is either ALL of the following or NONE:

1. a basis of V
2. a minimal spanning set of V
3. a maximal LI set in V
4. an independent set in V, where $\mathrm{Dim}(V) = n$
5. a spanning set for V, with $\mathrm{Dim}(V) = n$

These results can save a lot of time and effort.

Example: Is the following space equal to $\mathbb{R}^3$?

$$\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Back in the day we'd have to compare this to $[x, y, z]^T$, an arbitrary vector. Painful. However, it's a 3 dimensional space, there are three vectors, we need only confirm that they're independent. That's not easy, but it's not that hard.

$$a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so we get the equations

$$a + 2c = 0 \qquad 2b + c = 0 \qquad -a + b - c = 0$$

so $c = -2b$ and $a = 4b$ which, in the last equation, produces $-4b + b + 2b = -b = 0$, so $a = b = c = 0$ and the set is linearly independent. In a three dimensional space any three linearly independent vectors form a basis, so done, they're a spanning set.

Dimension and Subspaces

Subspaces work exactly the way you would expect with bases and dimension.

Proposition: If U is a subspace of V then $\mathrm{Dim}(U) \leq \mathrm{Dim}(V)$.

Proof.
This one is actually fairly simple. Any basis of U is also a linearly independent set in V, and so must have fewer or the same number of elements as a basis of V. ■

Proposition: If U is a subspace of V and $\mathrm{Dim}(U) = \mathrm{Dim}(V)$ then $U = V$.

Proof.
Also fairly simple. If U has the same dimension as V then a basis of U must have as many elements as a basis of V, and be linearly independent. It becomes a maximal independent set in V and so is a basis. ■

The Main Point of Bases:

Theorem:
Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for vector space V. For any $\mathbf{x} \in V$ there are *unique* coefficients $a_1, \ldots, a_n$ such that

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Proof.
First, we can get some values a since the vectors span V. Now we need them to be unique. Assume we have another set of values:

$$\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$$

so

$$\mathbf{x} - \mathbf{x} = \mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$$

which is a linear combination for the zero vector, and the set of vectors is linearly independent. As a result, each coefficient $(a_k - b_k) = 0$, so the a and b are equal. The expansions are unique. ■

Example:

Take the vector $[1, 3, 0]$ and write it in terms of the standard $\mathbb{R}^3$ basis:

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Next, we'll use

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

We need

$$a \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

The last line shows $a = -b$, which combines with the second line for $c = 3$. Then $b = -1$ and $a = 1$. Final answer:

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Some Basis Related Questions:

We haven't really done enough on this subject yet, so here we go.

If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ form a basis of vector space V then: does the set $\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$ form a basis of V as well?

Not as hard as it looks. All you have to do is apply the standard information on bases, the definitions, the rules, etc. First: V is a vector space which has a 3 sized basis. Therefore, it is 3 dimensional. As a result, a set, like the one we're given, spans V if and only if it is linearly independent. So, we need only check linear independence.

$$a(\mathbf{x}+\mathbf{y}) + b(\mathbf{y}+\mathbf{z}) + c(\mathbf{z}+\mathbf{x}) = \mathbf{0}$$
$$(a+c)\mathbf{x} + (a+b)\mathbf{y} + (b+c)\mathbf{z} = \mathbf{0}.$$

That is a linear combination. Of three Linearly Independent vectors. That means the three coefficients (scalar multipliers, whatever) HAVE to be zero. So:

$$a+c=0 \qquad a+b=0 \qquad b+c=0.$$

That means $c = -a$, $b = -a$. Using those on the last equation gives us $-2a = 0$ so $a = 0$ then $a = b = c = 0$. So, that means that

$$a(\mathbf{x}+\mathbf{y}) + b(\mathbf{y}+\mathbf{z}) + c(\mathbf{z}+\mathbf{x}) = \mathbf{0} \qquad \Longrightarrow \qquad a = b = c = 0$$

and the three vectors are LI, so they are a basis.

Now the same for $\{\mathbf{x}-\mathbf{y}, \mathbf{y}-\mathbf{z}, \mathbf{z}-\mathbf{y}\}$.

$$a(\mathbf{x}-\mathbf{y})+b(\mathbf{y}-\mathbf{z})+c(\mathbf{z}-\mathbf{x})=\mathbf{0}$$

leads to

$$(a-c)\mathbf{x}+(b-a)\mathbf{y}+(c-b)\mathbf{z}=\mathbf{0}$$

so

$$a-c=0 \qquad b-a=0 \qquad c-b=0.$$

We get $c=a$ and $b=a$. Substituting those into the last equation results in $a-a=0$ which is true. So, we get $a=b=c=$free. Not just zero, so those three vectors are linearly dependent and not a basis.

Reducing a Spanning Set:

We do not want to do this too often, but lets take a look:

$$\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

is a spanning set of $\mathbb{R}^3$. Lets reduce it to a basis. This may take a while.

$$a\begin{bmatrix}1\\2\\-1\end{bmatrix}+b\begin{bmatrix}0\\2\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix}+f\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

for

$$a+c+f=0 \qquad 2a+2b+c+d=0 \qquad -a+b-d+f=0.$$

$a=-c-f$. Substitute that into the other equations for

$$a=-c-f \qquad 2b-c+d-2f=0 \qquad b+c-d+2f=0$$

and $b=-c+d-2f$. Substitute that into the middle equation for

$$a=-c-f \qquad -3c+3d-6f=0 \qquad b=-c+d-2f.$$

Finally, we get $c=d-2f$, which we can substitute in for

$$a=-d+f \qquad c=d-2f \qquad b=0.$$

So, we can get rid of the vectors corresponding to a, c, d, f, NOT b, since it came out to zero, it is NOT variable by this solution (which happens to be correct).

So, lets ditch the second last one.

$$\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

and then look into

$$a\begin{bmatrix}1\\2\\-1\end{bmatrix}+b\begin{bmatrix}0\\2\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

So,

$$a+c+d=0 \qquad 2a+2b+c=0 \qquad -a+b+d=0$$

so $a=-c-d$, and

$$a=-c-d \qquad 2b-c-2d=0 \qquad b+c+2d=0$$

and $b=-c-2d$ for

$$a=-c-d \qquad -3c-6d=0 \qquad b=-c-2d$$

so $c=-2d$ so

$$a=d \qquad c=-2d \qquad b=0$$

what a surprise. Anyway, get rid of a vector associated with a,c,d and you're done. Cut the last one for

$$\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}.$$

For exercises, see the previous section. Any additional exercises will be posted as announcements on virtual campus.